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Spontaneous pair production in a magnetic monopole field

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Abstract. Electrically charged scalar particle–antiparticle pair creation in the field of a magnetic monopole is studied by path integrals. It is found that a magnetic charge g creates pairs with charge e, with angular momentum $l < eg - \frac{1}{2}$ with the asimuthal component $m = 0$.

1. Introduction

The Coulomb field of a charge exceeding the value 137/2 spontaneously produces pairs of particles and antiparticles [1]. A mathematical explanation is as follows. The state functions describing the charge particles moving in the Coulomb field are given in terms of the Whittaker functions $M_{K,\mu}(\rho)$. Here the index K is always positive and the argument ρ is proportional to the radial coordinate r. The second index which is important for our discussion is given by $\mu = \sqrt{(l+\frac{1}{2})^2 - e^2Q^2}$ where l is the angular momentum and Q is the charge of the source. For the supercritical sources, μ becomes imaginary and we cannot distinguish $M_{K,\mu}$ and its complex conjugate $M^*_{K,\mu} = M_{K,-\mu}$ on the basis of finiteness at the origin. Thus for small values of the angular momentum we always have the mixture of two solutions which are complex conjugates of each other. The magnitude of this mixture can be found by imposing the condition that the solution should be finite for r going to infinity. The unique solution satisfying this condition can be given in terms of the other Whittaker function, $W_{K,\mu}$. When we express $W_{K,\mu}$ in terms of $M_{K,\mu}$ and $M^*_{K,\mu}$, we find the relative coefficient between these two functions which gives the pair-production amplitude.

In this work we investigate the possibility of electrically-charged pair production in the field of a magnetic monopole. The situation is similar to the Coulomb case: we observe that a magnetic monopole g always produces the charged pairs with angular momentum constrained to the values $(l+\frac{1}{2})^2 < e^2g^2$. We adopt the method of path integrations, which has proved to be very convenient in pair production studies in cosmological backgrounds as well as in general electromagnetic potentials [2–4].

2. The propagator

The propagator for a charged scalar particle of mass μ coupled to the external electromagnetic field A_ν is given by [1] (with $\hbar = 1$)

$$G(x_a, x_b) = \int_0^\infty \mathrm{d}W\, \mathrm{e}^{-\mathrm{i}\mu^2 W} \int \mathrm{D}^4x \exp\left[\frac{\mathrm{i}}{4}\int_0^W \mathrm{d}w(-\dot{t}^2 + \dot{\boldsymbol{x}}^2 + 4eA^\nu \dot{x}_\nu)\right] \tag{1}$$

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which can also be expressed in the phase space form as [4]

$$G(x_a, x_b) = \int_0^\infty \mathrm{d}W\, \mathrm{e}^{-\mathrm{i}\mu^2 W} \int \mathrm{D}^4x\mathrm{D}^4p \exp\left[\mathrm{i}\int_0^W \mathrm{d}w(p_0\dot{t} + \boldsymbol{p}\cdot\dot{\boldsymbol{x}} + p_0^2 - \boldsymbol{p}^2 + eA\dot{x})\right] \quad (2)$$

where the overdots stand for derivatives with respect to the 'parameter time' w. The above path integral is understood as the usual graded formulation [3]

$$G(x_a, x_b) = \int_0^\infty \mathrm{d}W\, \mathrm{e}^{-\mathrm{i}\mu^2 W} \lim_{\substack{n\to\infty \\ \epsilon\to 0}} \prod_{j=1}^{n}\int_{-\infty}^{\infty} \mathrm{d}^4x_j \prod_{j=1}^{n+1}\int_{-\infty}^{\infty} \frac{\mathrm{d}^4p_j}{(2\pi)^4} \exp[\mathrm{i}p_{0j}(t_j - t_{j-1})$$

$$+\mathrm{i}\boldsymbol{p}_j\cdot(\boldsymbol{x}_j - \boldsymbol{x}_{j-1}) + \mathrm{i}\epsilon p_{0j}^2 - \mathrm{i}\epsilon \boldsymbol{p}_j^2 + \mathrm{i}eA_j(x_j - x_{j-1})]$$

with

$$x_a = x_0 \qquad x_b = x_{n+1} \qquad W = (n+1)\epsilon.$$

For the magnetic monopole g, located at the origin, A_ν can be expressed in terms of spherical coordinates, as $[\hbar = c = 1]$

$$A_\nu = \frac{g(1-\cos\theta)}{r\sin\theta}\hat{\varphi}. \quad (3)$$

Inserting this potential into (2) and adopting the spherical coordinates we have

$$G(x_a, x_b) = \int_0^\infty \mathrm{d}W\, \mathrm{e}^{-\mathrm{i}\mu^2 W} \int \mathrm{D}t\mathrm{D}p_0 \exp\left[\mathrm{i}\int_0^W \mathrm{d}w(p_0\dot{t} + p_0^2)\right]$$

$$\times \int \mathrm{D}\boldsymbol{x}\mathrm{D}\boldsymbol{p} \exp\left[\mathrm{i}\int_0^W \mathrm{d}w\left(p_r\dot{r} + p_\theta\dot{\theta} + p_\varphi\dot{\varphi} - p_r^2 - \frac{p_\theta^2 - 1/4}{r^2}\right.\right.$$

$$\left.\left. - \frac{p_\varphi^2 - 1/4}{r^2\sin^2\theta} + eg(1-\cos\theta)\dot{\varphi}\right)\right]. \quad (4)$$

The terms $1/4r^2$ and $1/4r^2\sin^2\theta$ are the usual ordering terms [5]. The form of the interaction term suggests a translation of p_φ by $p_\varphi \to p_\varphi - eg(1-\cos\theta)$; which leads to

$$G(x_a, x_b) = \int_0^\infty \mathrm{d}W\, \mathrm{e}^{-\mathrm{i}\mu^2 W} \int \mathrm{D}t\mathrm{D}p_0 \exp\left[\mathrm{i}\int_0^W \mathrm{d}w(p_0\dot{t} + p_0^2)\right]$$

$$\times \int \mathrm{D}\boldsymbol{x}\mathrm{D}\boldsymbol{p} \exp\left[\mathrm{i}\int_0^W \mathrm{d}w\left(p_r\dot{r} + p_\theta\dot{\theta} + p_\varphi\dot{\varphi} - p_r^2 - \frac{p_\theta^2 - e^2g^2 - 1/4}{r^2}\right.\right.$$

$$\left.\left. - \frac{p_\theta^2 + 2eg(p_\varphi - eg)\cos\theta - 2eg(p_\varphi - eg) - 1/4}{r^2\sin^2\theta}\right)\right]. \quad (5)$$

There are several recipes in the literature on the derivation of the ordering terms resulting from the point canonical transformations in path integrals. Since we are not studying path integral problems in the present paper, we will not discuss these recipes here. We are merely interested in deriving the correct Green's function for our problem. By writing the covariant Schrödinger equation in (x_μ, w), $(\mu = 0, 1, 2, 3)$ 'spacetime' and performing the transformations to the polar coordinates we can actually check that the ordering terms in (5) are correct.

The trivial path integrals over $\mathrm{D}t\mathrm{D}p_0$ and $\mathrm{D}\varphi\mathrm{D}p_\varphi$ fix the values $p_0 = E$ and $p_\varphi = m = 0, 1, 2, \ldots$. Then the expression for the propagator reads

$$G(x_a, x_b) = \int_0^\infty \mathrm{d}W \sum_{m=-\infty}^{\infty} \int \frac{\mathrm{d}E}{2\pi} \mathrm{e}^{\mathrm{i}E(t_b - t_a)} \mathrm{e}^{\mathrm{i}m(\varphi_b - \varphi_a)} \mathrm{e}^{\mathrm{i}(E^2 - \mu^2)W} F(r, \theta; W) \quad (6)$$

where

$$F(r,\theta;W) = (r_a^2 r_b^2 \sin\theta_a \sin\theta_b)^{-1/2} \times \int \mathrm{D}r\mathrm{D}p_r\mathrm{D}\theta\mathrm{D}p_\theta \exp\left[\mathrm{i}\int_0^W \mathrm{d}w\left(p_r\dot{r} + p_\theta\dot{\theta} - p_r^2 - \frac{p_\theta^2 - e^2g^2 - 1/4}{r^2} - \frac{m^2 + 2eg(m-eg)\cos\theta - 2eg(m-eg) - 1/4}{r^2\sin^2\theta}\right)\right]. \tag{7}$$

The dynamics of the θ coordinate is of the Pöschl–Teller potential type and its exact solution is known [6]:

$$F(r,\theta;W) = \sum_{l=0}^{\infty} P_{\alpha\beta}^l(\cos\theta_a) P_{\alpha\beta}^l(\cos\theta_b)\bar{F}(r;W). \tag{8}$$

Here, $P_{\alpha\beta}^l(\cos\theta)$ are the well known $SU(2)$ matrix elements which are related to the Jacobi functions. The indices are defined as

$$\alpha = m - eg \qquad \beta = eg. \tag{9}$$

$\bar{F}$ is the radial path integral given by

$$\bar{F} = \frac{1}{r_a r_b}\int \mathrm{D}r\mathrm{D}p_r \exp\left[\mathrm{i}\int_0^W \mathrm{d}w\left(p_r\dot{r} - p_r^2 - \frac{(l+1/2)^2 - e^2g^2 - 1/4}{r^2}\right)\right] \tag{10}$$

whose solution is

$$\bar{F} = \frac{1}{r_a r_b}\frac{-\mathrm{i}}{W}\exp\left[\frac{\mathrm{i}}{4W}(r_a^2 + r_a^2)\right] I_\gamma\left(\frac{r_a r_b}{2\mathrm{i}W}\right) \tag{11}$$

where

$$\gamma = \sqrt{(l+1/2)^2 - e^2g^2}. \tag{12}$$

After introducing (11) and (8) into (6) and performing the integration over $\mathrm{d}W$ we obtain

$$G(x_a,x_b) = \int \frac{\mathrm{d}E}{2\pi}\mathrm{e}^{\mathrm{i}E(t_b-t_a)}\sum_{l,m} G_{Elm}(x_a,x_b). \tag{13}$$

Here, G_{Elm} is the propagator for the mode E, l, m which (with $r_a > r_b$) is given by

$$G_{Elm}(x_a,x_b) = \frac{1}{4}\mathrm{e}^{\mathrm{i}m(\varphi_b-\varphi_a)} P_{\alpha\beta}^l(\cos\theta_a) P_{\alpha\beta}^l(\cos\theta_b)\frac{1}{\sqrt{r_a r_b}} H_\gamma^{(2)}\left(\sqrt{E^2-\mu^2}\,r_a\right) \times J_\gamma\left(\sqrt{E^2-\mu^2}\,r_b\right) \tag{14}$$

where $H_\gamma^{(2)}$ and J_γ are the Bessel functions.

3. Wavefunctions

Wavefunctions are the solutions of the Klein–Gordon equation

$$[(\mathrm{i}\partial_\mu - eA_\mu)^2 - \mu^2]f(x) = 0 \tag{15}$$

with A_μ given by (3). In spherical coordinates the above equation is solved by

$$f(x) = \mathrm{e}^{-\mathrm{i}Et}\frac{\mathrm{e}^{-\mathrm{i}m\varphi}}{\sqrt{2\pi}}\sqrt{l+1/2}\,P_{\alpha\beta}^l(\cos\theta)R(r). \tag{16}$$

The radial wavefunction R is given either by

$$R_{>}(r) = N_{>}\frac{1}{\sqrt{r}}H_{\gamma}^{(2)}\left(\sqrt{E^2-\mu^2}r\right) \tag{17}$$

or

$$R_{<}(r) = N_{<}\frac{1}{\sqrt{r}}J_{\gamma}\left(\sqrt{E^2-\mu^2}r\right) \tag{18}$$

which are regular at $r \to \infty$ or at $r \to 0$, respectively.

We normalize $R_{>}$ and $R_{<}$ on the timelike surfaces defined by constant r in the limit $r \to \infty$ and $r \to 0$, respectively. From the requirements

$$R_{>}(r)\partial_r R_{>}^{*}(r)\mid_{r\to\infty}= 1 \tag{19}$$

and

$$R_{<}(r)\partial_r R_{<}^{*}(r)\mid_{r=0}= 1 \tag{20}$$

we obtain

$$N_{>} = \frac{\sqrt{\pi}}{2}\mathrm{e}^{-\pi\bar{\gamma}/2} \tag{21}$$

and

$$N_{<} = \frac{1}{\sqrt{\bar{\gamma}}}\Gamma(\mathrm{i}\bar{\gamma}+1) \tag{22}$$

where

$$\bar{\gamma} \equiv \mathrm{i}\gamma = \sqrt{e^2g^2-(l+1/2)^2}. \tag{23}$$

4. The amplitude

The amplitude for detecting a particle and an antiparticle at the timelike surfaces r_a and r_b is given by [2]

$$A = A_0\int \mathrm{d}E_a \int \mathrm{d}E_b \sum_{(l,m)_{a,b}} \int \mathrm{d}\sigma_a^{\mu}\,\mathrm{d}\sigma_b^{\nu}\, f^{*}(a)\,\overleftrightarrow{\partial}_{\mu_a} f^{*}(b)\,\overleftrightarrow{\partial}_{\nu_b} G(x_a,x_b). \tag{24}$$

Here, $\mathrm{d}\sigma^{\mu}$ is the element of the spherical surface with radius r and μ is the radial direction from the surface:

$$\mathrm{d}\sigma^{\mu} = r^2\,\mathrm{d}t\,\mathrm{d}\theta\,\sin\theta\,\mathrm{d}\varphi.$$

A_0 is the vacuum persistence amplitude.

We insert (14) into (24) and use (17) and (18) for $f(a)$ and $f(b)$, respectively. Integrations over $\mathrm{d}t$ and $\mathrm{d}\varphi$ at points a and b give

$$E_a = -E_b \equiv E \tag{25}$$

and

$$m_a = -m_b \equiv m. \tag{26}$$

Integrals over $\mathrm{d}\theta_a$ and $\mathrm{d}\theta_b$, by the virtue of the orthonormality of $P_{\alpha\beta}^{l}$ functions [7] give

$$l_a = l_b \equiv l \qquad \alpha_a = \alpha_b \equiv \alpha \qquad \beta_a = \beta_b \equiv \beta. \tag{27}$$

Equation (26) and the condition on α's in equation (27) imply

$$m_a = m_b = 0. \tag{28}$$

Thus only the pairs with modes $m = 0$ are produced.

Taking into account the above considerations, the amplitude becomes

$$A = \sum_l \int \mathrm{d}E \, A_{Elm} \tag{29}$$

where A_{Elm} is the amplitude for creation of a pair with quantum numbers $E, -E; l$ and $m = 0$:

$$\begin{aligned} A_{Elm} &= A_0 \delta_{m,0} r_a^2 r_b^2 R^*_{Elm}(a) \overleftrightarrow{\partial}_{r_a} R^*_{-E,l,-m}(b) \overleftrightarrow{\partial}_{r_b} G_{Elm}(r_a r_b) \\ &= A_0 \delta_{m,0} r_a^2 r_b^2 [\dot{R}^*_a \dot{R}^*_b - (\dot{R}^*_a R^*_b \partial_{r_b} + R^*_a \dot{R}^*_b \partial_{r_a}) + R^*_a R^*_b \partial_{r_a} \partial_{r_b}] G_{Elm}(r_a r_b). \end{aligned} \tag{30}$$

Here G is the radial part of the propagator (14):

$$G_{Elm}(r_a, r_b) = \frac{1}{4\sqrt{r_a r_b}} H^{(2)}_\gamma \left(\sqrt{E^2 - \mu^2} r_a \right) J_\gamma \left(\sqrt{E^2 - \mu^2} r_b \right). \tag{31}$$

For surfaces $r_a > r_b$ the radial wavefunctions R_a and R_b are given by (17) and (18) respectively.

We want to evaluate the amplitude of equation (30) in $r_a \to \infty$, $r_b \to 0$ limit. Two cases are distinguished.

(i) For the real values of γ, that is for $l + \frac{1}{2} \geqslant eg$, A_{Elm} vanishes.

(ii) For the imaginary values of γ, that is for $l + \frac{1}{2} < eg$ the amplitude is non-zero

$$A_{Elm} = A_0 2\sqrt{\pi} \frac{\sqrt{\bar{\gamma}} \mathrm{e}^{-\pi \bar{\gamma}/2}}{\Gamma(-\mathrm{i}\bar{\gamma} + 1)} \theta(eg - l - 1/2) \delta_{m,0}. \tag{32}$$

The probabilities for one and n pair production in the mode E, l, m are

$$P_1 = |A_{Elm}|^2 \equiv \theta(eg - l - 1/2) |A_0|^2 \delta_{m,0} (1 - \mathrm{e}^{-2\pi \bar{\gamma}}) \equiv |A_0|^2 \omega \tag{33}$$

and

$$P_n = |A_0|^2 \omega^n. \tag{34}$$

After calculating $|A_0|^2$ from the conservation of probability we have

$$P_n = \theta(eg - l - 1/2) \delta_{m,0} \omega^n (1 - \omega). \tag{35}$$

The average number of pairs is

$$\bar{n} = \sum_{n=0}^{\infty} n P_n = \frac{\omega}{1 - \omega} = \theta(eg - l - 1/2) \delta_{m,0} (\mathrm{e}^{2\pi \bar{\gamma}} - 1). \tag{36}$$

5. Conclusion

We found that in the field of a magnetic monopole g, pairs of electric charges e are created with total angular momenta

$$l < eg - 1/2 \qquad m = 0. \tag{37}$$

The situation is very similar to the charged pair creation in the Coulomb field exceeding $Q = 137/2$.

For $r \to 0$, the radial wavefunction describing the motion of an electrically-charged particle in the field of magnetic monopole is $R_< \simeq J_{\mp\gamma}(\sqrt{E^2 - \mu^2}) \simeq r^{\pm\gamma/2}$. For $\gamma = \sqrt{(l + \frac{1}{2})^2 - e^2 g^2}$ real and positive (or negative), the positive (or negative) sign for the index of the Bessel function is chosen to have finite solution at the origin. Thus for real γ we have a unique solution. However, for imaginary values of γ we do not have any physical criteria to distinguish between J_γ and $J_{-\gamma} = J_\gamma^*$. Therefore, recalling the relation between the Hankel and Bessel functions [8]

$$H_\gamma^{(2)} = \frac{1}{\sin \pi \bar{\gamma}} (J_\gamma e^{-\pi \bar{\gamma}} - J_{-\gamma}) \tag{38}$$

for imaginary γ's one can establish a correlation between the states describing the particles at $r \to \infty$ and $r \simeq 0$.

Wavefunctions describing the particles at $r \to \infty$ and $r \simeq 0$ given by

$$f_{Elm}^{>} \simeq e^{-iEt} \frac{e^{im\varphi}}{\sqrt{2\pi}} \sqrt{l + 1/2} P_{\alpha\beta}^{l}(\cos\theta) H_\gamma^{(2)} \left(\sqrt{E^2 - \mu^2} r\right) \tag{39}$$

$$f_{elm}^{<} \simeq e^{-iEt} \frac{e^{-im\varphi}}{\sqrt{2\pi}} \sqrt{l + 1/2} P_{\alpha\beta}^{l}(\cos\theta) J_\gamma^{(2)} \left(\sqrt{E^2 - \mu^2} r\right) \tag{40}$$

are related, for $m = 0$, as

$$f_{El0}^{>}(x) \simeq -\frac{1}{\sin \pi \bar{\gamma}} [e^{-\pi \bar{\gamma}} f_{El0}^{<}(x) - (f_{El0}^{<}(x))^*]. \tag{41}$$

This means that for certain values of quantum numbers (i.e. for $(l + \frac{1}{2})^2 < e^2 g^2$ and $m = 0$) the particle states at $r \to \infty$ are equal to the linear combination of the particle and antiparticle states of $r \simeq 0$. The asymptotic vacuum is equivalent to the vacuum at the origin only, with the probability $(e^{-\pi\gamma})^2$ or the relative probability for the pair creation given by (33)

$$\omega = 1 - e^{-2\pi\gamma}.$$

The magnitude of the elementary magnetic charge is $eg = \frac{1}{2}$ [9] or $eg = 1$ [10]. Thus, unlike the case of the Coulomb field for which a superheavy nucleus is required, to have charged pair production large quantities of magnetic charges are not needed. If magnetic monopoles exist in nature, an amount equal to twice the elementary value is enough to create electrically-charged particles.

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References

[1] See, for example, Lapedes A S 1970 *Phys. Rev.* D **17** 2556
[2] Hartle J B and Hawking S W 1976 *Phys. Rev.* D **13** 2188
[3] Duru I H and Ünal N 1986 *Phys. Rev.* D **34** 959
[4] Barut A O and Duru I H 1990 *Phys. Rev.* D **41** 1312
Duru I H 1991 *Vacuum Structure in Intense Fields, NATO ASI Series B255* ed H M Fried and B Müller (New York: Plenum) p 402
[5] See, for example, Duru I H and Ünal N 1988 *J. Phys. A: Math. Gen.* **21** 3389
[6] Duru I H 1984 *Phys. Rev.* D **30** 2121
[7] Vilenkin N Ja 1968 *Special Functions of the Theory of Group Representations* (Providence, RI: American Mathematical Society)
[8] Magnus W, Oberhettinger F and Soni R P 1966 *Formulas and Theorems for the Special Functions of Mathematical Physics* 3rd edn (Berlin: Springer)
[9] Dirac P A M 1931 *Proc. R. Soc.* A **133** 60
[10] Schwinger J 1969 *Science* **105** 757